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1.1 Invariance under rigid motions

The Laplace equation is invariant under rigid motions, which means that if u is a solution of the Laplace equations, $u \circ R$ is a solution of the Laplace equation as well, where R means a rigid motion. In fact, the following identity holds for rigid motions:

$$\Delta(u \circ R) = \Delta u \circ R.$$

It is easy to verify the case that R is a translations. Next we verify the case that R is a rotation. We let

$$(Rx)_i = \sum_j a_{ij} x_j.$$

Then $A = (a_{ij})$ satisfies that

$$AA^T = A^T A = I,$$

that is,

$$\sum_k a_{ik} a_{jk} = \delta_{ij} \text{ and } \sum_k a_{ki} a_{kj} = \delta_{ij}. \tag{1}$$

We have

$$(u \circ R)_i = \sum_k (u_k \circ R) a_{ki},$$

$$(u \circ R)_{ii} = \sum_{kl} (u_{kl} \circ R) a_{li} a_{ki}.$$

Hence

$$\Delta(u \circ R) = \sum_{ikl} (u_{kl} \circ R) a_{li} a_{ki}.$$

By (1),

$$\Delta(u \circ R) = \sum_{kl} (u_{kl} \circ R) \delta_{kl} = \Delta u \circ R.$$

1.2 Laplacians in spherical coordinates in 2D

Next we calculate the Laplacians in spherical coordinates in 2D. In 2D, we have the spherical coordinate transformation

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta. \end{cases}$$

The Jacobian is

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

The inverse is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}.$$

Therefore, by the chain rule,

$$\begin{cases} \partial_x = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta; \\ \partial_y = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta. \end{cases}$$

Then,

$$\begin{aligned} \Delta = \partial_x^2 + \partial_y^2 = {} & [\cos\theta\,\partial_r(\cos\theta\,\partial_r) + \sin\theta\,\partial_r(\sin\theta\,\partial_r)] \\ & + \left[\frac{\sin\theta}{r}\partial_\theta\left(\frac{\sin\theta}{r}\partial_\theta\right) + \frac{\cos\theta}{r}\partial_\theta\left(\frac{\cos\theta}{r}\partial_\theta\right)\right] \\ & + \left[-\cos\theta\,\partial_r\left(\frac{\sin\theta}{r}\partial_\theta\right) + \sin\theta\,\partial_r\left(\frac{\cos\theta}{r}\partial_\theta\right)\right] \\ & + \left[-\frac{\sin\theta}{r}\partial_\theta(\cos\theta\,\partial_r) + \frac{\cos\theta}{r}\partial_\theta(\sin\theta\,\partial_r)\right]. \end{aligned}$$

By utilizing the cancellations in each pair, we obtain that

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

1.3 Laplacians in spherical coordinates in 3D

Next we calculate the Laplacians in spherical coordinates in 3D. We will use cylindrical coordinates as an intermediate coordinate to convert the 3D cases to 2D cases. Suppose that the Cartesian coordinate is (x, y, z) and the spherical coordinate is (r, φ, θ):

$$\begin{cases} x = r\sin\theta\cos\varphi; \\ y = r\sin\theta\sin\varphi; \\ z = r\cos\theta. \end{cases}$$

Then the cylindrical coordinate (s, φ, z) is

$$\begin{cases} x = s\cos\varphi; \\ y = s\sin\varphi; \\ z = z. \end{cases}$$

We could observe that the transformations (x, y, z) to (s, φ, z) and (s, φ, z) to (r, φ, θ) are similar to the 2D spherical coordinate transformation. So we have

$$\begin{aligned} \Delta &= \partial_x^2 + \partial_y^2 + \partial_z^2 \\ &= \partial_s^2 + \frac{1}{s}\partial_s + \frac{1}{s^2}\partial_\varphi^2 + \partial_z^2 \\ &= \frac{1}{s}\partial_s + \frac{1}{s^2}\partial_\varphi^2 + \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2. \end{aligned}$$

So it remains to calculate ∂_s.

$$\partial_s = r_s\partial_r + \theta_s\partial_\theta.$$

Since

$$\begin{cases} z = r\cos\theta; \\ s = r\sin\theta, \end{cases}$$

$$\partial_s = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

Therefore,

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left(\partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi^2\right).$$

The contents of this notes are mainly from Strauss's PDE section 6.1.